§ 5.2 cont'd

note: appended to L16 (finished last problem)

Now that we've found an orthonormal basis of a plane, let's generalize further:

given: 3-dim subspace V of $\mathbb{R}^n$ with basis of $\vec{v}_1, \vec{v}_2, \vec{v}_3$

1^{st}, find an orthonormal basis $\vec{u}_1, \vec{u}_2$ on plane $E = \text{span}(\vec{v}_1, \vec{v}_2)$ (use steps from previous example)

2^{nd}, resolve $\vec{v}_3$ into its components $\parallel$ and $\perp$ to plane E:

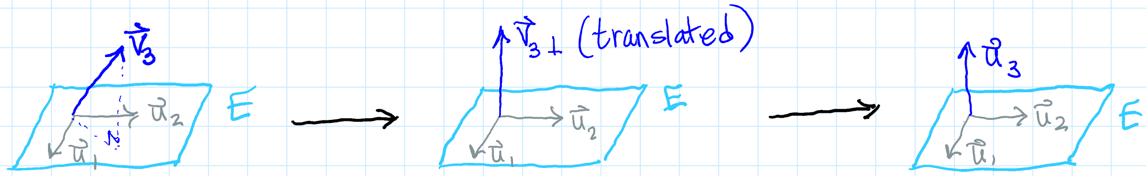
$$\vec{v}_3 = \vec{v}_{3\parallel} + \vec{v}_{3\perp}$$

$$\Rightarrow \vec{v}_{3\perp} = \vec{v}_3 - \vec{v}_{3\parallel}$$

$$= \vec{v}_3 - \text{proj}_E \vec{v}_3$$

$$\boxed{\vec{v}_{3\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2}$$

3^{rd}, determine $\vec{u}_3 = \dfrac{\vec{v}_{3\perp}}{\|\vec{v}_{3\perp}\|}$



to generalize further, use <u>Gram-Schmit Process</u>:

given basis $\vec{v}_1, \ldots, \vec{v}_m$ of subspace V of $\mathbb{R}^n$ for $j = 2, \ldots, m$

resolve $\vec{v}_j$ into its $\parallel$ and $\perp$ components to span preceding vectors $\vec{v}_1, \ldots, \vec{v}_{j-1}$

i.e., $\vec{v}_j = \vec{v}_{j\parallel} + \vec{v}_{j\perp}$ WRT $\text{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$

then
$$\left.\begin{aligned} \vec{u}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} \\ \vec{u}_2 &= \frac{\vec{v}_{2\perp}}{\|\vec{v}_{2\perp}\|} \\ &\vdots \end{aligned}\right\} \text{ is orthonormal basis of } V$$

$\|\vec{v}_{2\perp}\|$ is orthonormal basis of V

$\vdots$

$$\vec{u}_j = \frac{\vec{v}_{j\perp}}{\|\vec{v}_{j\perp}\|} \qquad \vec{v}_{j\perp} = \vec{v}_j - \vec{v}_{j\parallel} = \vec{v}_j - (\vec{u}_1\cdot\vec{v}_j)\vec{u} - \dots - (\vec{u}_{j-1}\cdot\vec{v}_j)\vec{u}_{j-1}$$

$\vdots$

$$\vec{u}_m = \frac{\vec{v}_{m\perp}}{\|\vec{v}_{m\perp}\|}$$

Section 5.3 Orthogonal Transformation and Matrices

- a LT, $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if it preserves vector length

  i.e., $\|T(\vec{x})\| = \|\vec{x}\| \quad \forall \vec{x} \in \mathbb{R}^n$

- if $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation then A is an orthogonal matrix

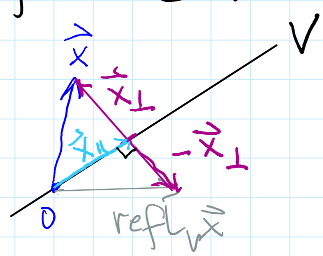
recall $\vec{v}_1 = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$ previously saw $\vec{v}_1 \cdot \vec{v}_2 = 0$ concluded $\vec{v}_1, \vec{v}_2$ are orthogonal

now, can say $T(\vec{x}) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\vec{x}$ is an orthogonal transformation $\mathbb{R}^2 \to \mathbb{R}^2$

and $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ is an orthogonal matrix for $\forall\theta$

ex. given subspace V of $\mathbb{R}^n$ for $\vec{x} \in \mathbb{R}^n$

$$\text{ref}_V \vec{x} = \vec{x}_\parallel - \vec{x}_\perp$$



using Pythagorean Theorem:

$$\|\vec{x}_\parallel\|^2 + \|-x_\perp\|^2 = \|\text{ref}_V\vec{x}\|^2$$

$$\Downarrow$$

$$\|\vec{x}_\parallel\|^2 + \|\vec{x}_\perp\|^2 = \|\vec{x}\|^2$$

$\|\text{ref}_V\vec{x}\| = \|\vec{x}\|$ (preserves length)

conclude: reflection is an orthogonal transformation

but wait, there's more ...

thm: given orthogonal transformation $T: \mathbb{R}^n \to \mathbb{R}^n$

if $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal

then $T(\vec{v})$, $T(\vec{w})$ ...

if $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal

then $T(\vec{v}), T(\vec{w})$ are orthogonal

i.e. orthogonal transformations preserve <u>right angles</u>

another way to think of this:

LT $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal iff $T(\vec{e}_1), T(\vec{e}_2) \dots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$

$n \times n$ matrix A is orthogonal iff its columns form an orthonormal basis of $\mathbb{R}^n$

recall: LT $T: \mathbb{R}^m \to \mathbb{R}^n$ then $T(\vec{x}) = A\vec{x}$

with $A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix}$ where $e_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ ith entry

ex. show $A = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal

$\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4$

sol'n: show columns of A form an <u>orthonormal</u> basis of $\mathbb{R}^4$

1. show $\|\vec{v}_1\| = \dots \|\vec{v}_4\| = 1$
2. show $\vec{v}_1 \cdot \vec{v}_2 = 0, \dots \vec{v}_3 \cdot \vec{v}_4 = 0$

<u>Transpose a Matrix</u>

ex. given an orthogonal $A = \frac{1}{7}\begin{bmatrix} 2^{11} & 6^{12} & 3^{13} \\ 3^{21} & 2^{22} & -6^{23} \\ 6^{31} & -3^{32} & 2^{33} \end{bmatrix} \leftarrow 3\times3$ $i \ j$

switching rows $\Leftrightarrow$ columns

$j \ i$
3×3

create B s.t. ij-th entry is ji-th entry of A

$B = \frac{1}{7}\begin{bmatrix} 2^{11} & 3^{21} & 6^{31} \\ 6^{12} & 2^{22} & -3^{32} \\ 3^{13} & -6^{23} & 2^{33} \end{bmatrix}$ refering components/entries of A

then $BA = \frac{1}{7} \cdot \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}$

BA's col 1: $\underbrace{\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}}_{B}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} \leftarrow$ 1st col. of A

$= \frac{1}{49}\begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = I_3$

$$= \frac{1}{49}\begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = I_3$$

$\begin{bmatrix} 3 & -6 & 2 \end{bmatrix}\begin{bmatrix} 6 \end{bmatrix}$

B ↑ 1st col. of A

$$= \begin{bmatrix} 4+9+36 \\ 12+6-18 \\ 6-18+12 \end{bmatrix} = \begin{bmatrix} 49 \\ 0 \\ 0 \end{bmatrix}$$

why does this work?

dot product

ij-th entry of BA is the [ith row of B] $\cdot$ [jth row of A]

since A is orthogonal, by the def'n of dot product,

product is 1 if $i = j$,
product is 0 if $i \neq j$

def'n: given $m \times n$ matrix A
then transpose of A or A^T is $n \times m$ matrix s.t.
its ijth entry is the jith entry of A
i.e. rows and columns are reversed

also, $n \times n$ matrix A is considered symmetric if $A^T = A$
and " " skew-symmetric if $A^T = -A$

ex. given $A = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 7 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 9 \\ 2 & 7 \\ 3 & 5 \end{bmatrix}$

Symmetric 2×2 matrices are of form $A = A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

ex. given $\underset{A^T}{\underset{\parallel}{A}} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ ↑ symmetric, then symmetric 2×2 matrices form a 3-dim subspace of $\mathbb{R}^{2\times 2}$ with basis $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

In contrast, skew-symmetric 2×2 matrices are of form $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$

ex. $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ is skew symmetric

$A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -A$

these would form a 1-dim. subspace with basis $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

if $a \neq 0$, $c \neq 0$: $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \neq A^T \neq A^{-T}$